McGILL UNIVERSITY

FACULTY OF ENGINEERING

FINAL EXAMINATION

MATH 263

ORDINARY DIFFERENTIAL EQUATIONS FOR ENGINEERS

Examiner: Professor J.J. Xu
Associate Examiner: Dr. R. Calleja Castillo

Date: Tuesday December 14, 2010
Time: 2:00 PM- 5:00 PM

INSTRUCTIONS

1. Please attempt to answer all questions in the exam booklets provided.
2. This exam is a total of 8 pints
3. This is a closed book examination. No books, crib sheets or lecture notes permitted.
4. Non-programmable and non-graphical calculators are permitted.
5. Dictionaries are not permitted.

This exam comprises the cover page, 3 pages of 8 questions and a page of the Table of Laplace Transforms.

Ordinary Differential Equations for Engineers MATH 263
Final Examination - December 14, 2010

1. (10 points) Find the general solution for each of the following first order differential equations:

   (a)
$$y' + \frac{y}{x(1+x)} = x^2(1+x). \tag{1}$$

   (b)
$$y' = y(3 - xy). \tag{2}$$

2. (10 points)

   (a) Find the general solution to the ODE $xy' + y = \cos(x)$.

   (b) Suppose that y_1 and y_2 are solutions of a first order linear ODE, and that $y_1 \neq y_2$. Write down a nonzero solution to the associated homogeneous linear ODE (in terms of y_1 and y_2). Then write down the general solution to the original equation (in terms of y_1 and y_2).

3. (10 points) Consider the linear operator

$$L = D^2 + cD + kI$$

   (a) For what values of c and k does the operator L have unit impulse response,

$$L\{w(t)\} = \delta(t), \quad y(0) = 0, y'(0) = 0,$$

   given by $w(t) = e^{-2t}\sin(t)$ for $t > 0$?

   (b) For this same operator L, write down the convolution integral for the solution to

$$L\{y\} = e^{-2t}, \quad y(0) = 0, y'(0) = 0,$$

   and evaluate it.

4. (10 points) A certain spring-mass system satisfies the initial value problem

$$y'' + 4y = \sin(t) - u_{2\pi}\sin(t - 2\pi), \quad y(0) = 0, y'(0) = 0.$$

(a) Find the solution for the initial value problem.

(b) Sketch the graph of the soluiton.

5. (10 points) Consider the ODE $y''' + 2y' = 2x$

(a) Find the annihilator Q(D).

(b) Find y_h and y_p using the annihilator method.

(c) Find y_p using y_h form (b) and the method of variation of parameters.

(d) Compare the particular solution y_p that you obtained in (c) with the y_p that you obtained in (b). They should be equal or differ by an element of y_h. Which method was more difficult? Explain.

6. (10 points) Consider the IVP

$$y'' - 3y' - 4y = 2e^{-t} \quad y(0) = \alpha, y'(0) = 0$$

(a) Find y_h.

(b) Seek a solution for the non-homogeneous equation of the form $Y(t) = v(t)y_1(t) = v(t)e^{-t}$. Substitute $Y(t)$ into the equation and find $v(t)$.

(c) Find y_p.

(d) Solve the initial value problem.

(e) Choose α so that the solutions tend to 0 as t tends to ∞.

7. (10 points) Consider the differential equation

$$(1 - x)y'' + y = 0.$$

(a) Seek power series solutions of the given differential equation about the point $x_0 = 0$. Find the recurrence relation.

(b) Find the first four terms not equal to zero in each of two solutions y_1 and y_2 (unless the series terminates sooner).

(c) By evaluating the Wronskian $W(y_1, y_2)(0)$ show that y_1 and y_2 form a fundamental set of solutions.

8. (10 points)The Legendre equation of order α is

$$(1 - t^2)u'' - 2tu' + \alpha(\alpha + 1)u = 0$$

(a) Find all the singular points and prove that they are regular.

(b) Consider the singular point $t_0 = 1$. Perform the change of variables $(t - 1) = x$ and show that the new equation is

$$(x^2 + 2x)y''(x) + 2(x + 1)y'(x) - \alpha(\alpha + 1)y(x) = 0$$

with $y(x) = u(x + 1)$. Show that this new equation has a regular singular point at $x_0 = 0$

(c) Find the indicial equation and the exponents at the singular point.

(d) Write down **the general form** of the two linearly independent solutions y_1 and y_2.